

# Electromagnetic instability and Schwinger effect in holographic QCD with a topological charge

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## Abstract

Using the D0-D4/D8 brane model holographically, we compute the vacuum decay rate for the Schwinger effect in the bubble and black brane configuration which corresponds to the confined (zero temperature) and deconfined (finite temperature) phase respectively in the large  $N_c$  QCD. Our calculation contains the influence of the D0-brane density which could be holographically related to  $\theta$  angle or chiral potential in QCD. Under the strong electromagnetic fields, the creation of a pair of quark-antiquark introduces the instability and the decay rate can be obtained by evaluating the imaginary part of the probe brane Lagrangian with a constant electromagnetic field. In the bubble D0-D4 configuration, we find the decay rate decreases when the  $\theta$  angle increases and it is consistent with the result in the black D0-D4 configuration at zero temperature limit. According to the calculations, we additionally find an instability from the vacuum of this system and interpret it as the description of the vacuum decay triggered by the  $\theta$  angle as it is known in the  $\theta$ -dependent QCD.

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# Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Review of the D0-D4/D8 model</b>	<b>4</b>
<b>3</b>	<b>Euler-Heisenberg Lagrangian of the D0-D4/D8 brane system</b>	<b>7</b>
3.1	Bubble D0-D4 geometry . . . . .	7
3.2	Black D0-D4 geometry . . . . .	9
<b>4</b>	<b>Holographic pair creation of quark-antiquark</b>	<b>11</b>
4.1	Imaginary part of the effective action at zero temperature . . . . .	11
4.2	Imaginary part of the action at finite temperature . . . . .	17
<b>5</b>	<b>Summary</b>	<b>20</b>

## 1 Introduction

Recent years, there have been many advances in the research on the strong electromagnetic field, especially in heavy-ion collision since it is expected that an extremely strong magnetic field is generated by the collision of the charged particles. Particularly, the Schwinger effect should be one of the most interesting phenomena in the heavy-ion collision, because the pair creation of charged particles from the vacuum occurs under such an externally strong electromagnetic field. In the Schwinger effect, the creation rate of a pair of charged particles could be obtained by evaluating the imaginary part of Euler-Heisenberg Lagrangian [1, 2]. However, the result implies the Schwinger effect is a non-perturbative effect which shows up only under the strong electromagnetic field.

Although it is still challenging to evaluate the Schwinger effect in strong electromagnetic field, the framework of gauge/gravity duality or AdS/CFT provides a powerful tool on studying the strongly coupled quantum field theories [3, 4, 5]. It has been recognized that a (d+2)-dimensional classical gravity theory could correspond to a (d+1)-dimensional gauge theory as a weak/strong duality. So with this framework, various applications of studying the Schwinger effect holographically have been presented [6, 7, 8, 9, 10, 11, 12, 13, 14]. Moreover, it turns out that in the top-down holographic approach, the imaginary part of the probe brane action could be identified as the creation rate of quark-antiquark pairs (i.e. the vacuum decay rate) [12, 13, 14]. While this is a different method, it allows us to quantitatively evaluate the effective Euler-Heisenberg Lagrangian in order to explore the electromagnetic instability holographically.

On the other hand, the  $\theta$ -dependence in QCD or Yang-Mills theory is also interesting.  $\theta$ -dependent gauge theories contain a Chern-Simons term as a topological term additional to the

action and the coupling of the Chern-Simons term is named as the  $\theta$  angle (as the following form),

$$S = \frac{1}{2g_{YM}^2} \int \text{Tr} F \wedge^* F - i \frac{\theta}{16\pi^2} \int \text{Tr} F \wedge F, \quad (1.1)$$

where  $g_{YM}$  is the Yang-Mills coupling constant and  $F$  is the gauge field strength. While the experimental value of  $\theta$  is small, the Chern-Simons term leads to many observable phenomena such as chiral anomaly, decay of the  $\theta$  vacuum, chiral magnetic effect and some effects in glueball condensation. Accordingly, to investigate the Schwinger effect with  $\theta$ -dependence in QCD would be significant since the creation rate of quark-antiquark pairs is affected by the  $\theta$  angle. So in this paper, we are motivated to study the electromagnetic instability and the Schwinger effect with such a topological term holographically.

In the gauge/gravity duality, the  $\theta$ -dependence could be introduced as the D-brane with D-instanton configuration in the string theory [15]. So for the method of QCD, we use the D0-D4/D8 system for our investigation since this system is holographically dual to QCD with a Chern-Simons term [16, 17] (See more details and applications in [18, 19, 20, 21]). In this model, the background geometry is produced by  $N_c$  coincident D4-branes wrapped on a cycle with  $N_0$  smeared D0-branes inside their worldvolume. The supersymmetry is broken down by imposing the anti-periodic boundary condition on fermions. Because the presence of the D0-branes, the effective action of the D4-branes take the following form,

$$S_{D4} = -\mu_4 \text{Tr} \int d^4 x dx^4 e^{-\phi} \sqrt{-\det(\mathcal{G} + \mathcal{F})} + \mu_4 \int C_5 + \frac{1}{2} \mu_4 \int C_1 \wedge \mathcal{F} \wedge \mathcal{F}, \quad (1.2)$$

where  $\mu_4 = (2\pi)^{-4} l_s^{-5}$ ,  $l_s$  is the length of the string,  $\mathcal{G}$  is the induced metric on the worldvolume.  $\mathcal{F} = 2\pi\alpha' F$  is the gauge field strength on the D4-brane.  $C_5$ ,  $C_1$  is the Romand-Romand 5- and 1- form respectively. We have used  $x^4$  to represent the wrapped direction which is periodic. The first term in (1.2) is the Dirac-Born-Infeld (DBI) action and the Yang-Mills action comes from its leading-order expansion by small  $\mathcal{F}$ . In the bubble D0-D4 solution, we have  $C_1 \sim \theta dx^4$  [17], thus D0-branes are actually D-instantons and the last term in (2.1) could be integrated as,

$$\int_{S_{x^4}} C_1 \sim \theta, \quad \int_{S_{x^4} \times \mathbb{R}^4} C_1 \wedge \mathcal{F} \wedge \mathcal{F} \sim \theta \int_{\mathbb{R}^4} \mathcal{F} \wedge \mathcal{F}. \quad (1.3)$$

Consequently, the number density of D0-branes (D0 charge) is related to the  $\theta$  angle and we could finally obtain the Yang-Mills plus Chern-Simons action (1.1) from (1.2) and (1.3) as the low-energy theory of the bubble D0-D4 system. As the analysis of the D4/D8 model (the Witten-Sakai-Sugimoto modle) [22, 23, 24, 25, 26], the bubble D0-D4 corresponds to the confinement phase of the dual field theory while the black D0-D4 corresponds to the deconfinement phase<sup>4</sup>. However in the black D0-D4 configuration, the physical interpretation of D0-brane is less clear

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<sup>4</sup>In [27], an alternative solution for the deconfinement phase has been proposed .

since D0-brane is not D-instanton. Nevertheless, we might identify the D0 charge as the chiral potential in the black D0-D4 configuration according to some evidences presented in [28, 29, 30]. Besides, the flavors can be introduced by a stack of  $N_f$  D8 and anti-D8 branes (D8/ $\overline{\text{D8}}$ -branes) as probes into the D0-D4 background. Hence the chirally symmetric or broken phase of the dual field theory is represented by the various configurations of D8/ $\overline{\text{D8}}$ -branes in the bubble or black D0-D4 background.

In this paper, we will focus on the derivation of the effective Euler-Heisenberg Lagrangian first, then we could explore the electromagnetic instability and evaluate the creation rate of quark-antiquark pairs in the vacuum. The paper is organized as follows, in section 2, we review the D0-D4/D8 system with more details. In section 3, we derive the the effective Euler-Heisenberg Lagrangian from the probe D8/ $\overline{\text{D8}}$ -branes action in the bubble and black D0-D4 background respectively. In section 4, we evaluate the creation rate of quark-antiquark pairs in the vacuum. In the black D0-D4 background, we find the creation rate is finite at zero temperature limit which qualitatively coincides with the results from the bubble D0-D4 background. And our numerical calculation shows the the creation rate decreases when the D0 density ( $\theta$  angle) increases in the bubble D0-D4 case. The final section is the summary and discussion.

## 2 Review of the D0-D4/D8 model

In this section, let us briefly review the D0-D4/D8 system. In string frame, the background geometry described by the bubble solution of  $N_c$  D4 branes with smeared  $N_0$  D0 charges. The near horizon metric reads in type IIA supergravity [17, 18],

$$ds^2 = \left(\frac{U}{R}\right)^{3/2} \left[ H_0^{1/2} \eta_{\mu\nu} dX^\mu dX^\nu + H_0^{-1/2} f(U) (dX^4)^2 \right] + H_0^{1/2} \left(\frac{R}{U}\right)^{3/2} \left[ \frac{dU^2}{f(U)} + U^2 d\Omega_4^2 \right]. \quad (2.1)$$

The D0-branes are smeared in the  $X^i$ ,  $i = 1, 2, 3$  and  $X^4$ .  $N_c$  represents the number of colors. The dilaton, the field strength of the Ramond-Ramond field, the function  $f(U)$ ,  $H_0(U)$  and the radius  $R$  of the bulk are given as follows,

$$e^\phi = g_s \left(\frac{U}{R}\right)^{3/4} H_0^{3/4}, \quad F_2 = \frac{1}{\sqrt{2!}} \frac{\mathcal{A}}{U^4} \frac{1}{H_0^2} dU \wedge dX^4, \quad F_4 = \frac{1}{\sqrt{4!}} \mathcal{B} \epsilon_4, \quad f(U) \equiv 1 - \frac{U_{KK}^3}{U^3}, \\ R^3 \equiv \pi g_s N_c l_s^3, \quad H_0 = 1 + \frac{U_{Q_0}^3}{U^3}, \quad \mathcal{A} = \frac{(2\pi l_s)^7 g_s N_0}{\omega_4 V_4}, \quad \mathcal{B} = \frac{(2\pi l_s)^3 g_s N_c}{\omega_4}. \quad (2.2)$$

where  $\mathcal{A}$ ,  $\mathcal{B}$  are two integration constants given as,

$$\mathcal{A} = 3\sqrt{U_{Q_0}^3 (U_{Q_0}^3 + U_{KK}^3)}, \quad \mathcal{B} = 3\sqrt{U_{Q_4}^3 (U_{Q_4}^3 + U_{KK}^3)}. \quad (2.3)$$

We use  $U_{KK}$ ,  $g_s$ ,  $l_s$ ,  $V_4$  and  $\epsilon_4$  to represent the coordinate radius of the bottom of the bubble, the string coupling, the string length, the volume of the unit four sphere  $S^4$  and the volume form of the  $S^4$  respectively.  $\alpha'$  is defined as  $l_s^2 = \alpha'$ . The coordinate  $U$  is the holographic radial direction, and  $U \rightarrow \infty$  corresponds to the boundary of the bulk. Therefore coordinate  $U$  takes the values in the region  $U_{KK} \leq U \leq \infty$ . In order to avoid a possible singularity at  $U = U_{KK}$ , the coordinate  $U$  satisfies the following periodic boundary condition [16],

$$X^4 \sim X^4 + \delta X^4, \quad \delta X^4 = \frac{4\pi}{3} \frac{R^{3/2}}{U_{KK}^{1/2}} H_0^{1/2}(U_{KK}) = 2\pi R. \quad (2.4)$$

So the Kaluza-Klein mass parameter is obtained,

$$M_{KK} = \frac{2\pi}{\delta X^4} = \frac{3U_{KK}^{1/2}}{2R^{3/2}} H_0^{-1/2}(U_{KK}). \quad (2.5)$$

The gauge coupling  $g_{YM}$  at the cut-off scale  $M_{KK}$  in the 4-dimensional Yang-Mills theory is derived as  $g_{YM}^2 = (2\pi)^2 g_s l_s / \delta X^4$  from the 5-dimensional D4-brane compactified on  $S^1$ . Thus, according to the gauge/gravity duality and AdS/CFT dictionary, the relationship between the parameters  $R, U_{KK}, g_s, U_{Q_0}$  (in the gravity side) and the parameters  $M_{KK}, \lambda, N_c, H_0(U_{KK})$  expressed in QCD is given as,

$$R^3 = \frac{1}{2} \frac{\lambda l_s^2}{M_{KK}}, \quad U_{KK} = \frac{2}{9} \lambda M_{KK} l_s^2 H_{KK}, \quad g_s = \frac{1}{2\pi} \frac{\lambda}{M_{KK} N_c l_s}, \quad (2.6)$$

where  $H_{KK} \equiv H_0(U_{KK})$  and  $\lambda \equiv g_{YM}^2 N_c$  is the 4-dimensional 't Hooft coupling. Since one of the spatial coordinates, denoted by  $X^4$ , is compactified on  $S^1$ , the fermions (and other irrelevant fields) would be massive by imposing the anti-periodic boundary condition on the cycle  $S^1$ , thus they are decoupled from the low-energy theory. Accordingly, the effective theory consists of the Yang-Mills fields only.

There is an alternatively allowed solution for the D0-D4/D8 system which is the black brane solution. This solution could be obtained by interchanging the coordinate  $X^4$  and  $X^0$  in (2.1). So the metric reads [21, 30],

$$ds_{D4}^2 = \left(\frac{U}{R}\right)^{3/2} \left[ -H_0^{-1/2} f_T(U) dt^2 + H_0^{1/2} \delta_{ij} dX^i dX^j + H_0^{1/2} (dX^4)^2 \right] \\ + H_0^{1/2} \left(\frac{R}{U}\right)^{3/2} \left[ \frac{dU^2}{f_T(U)} + U^2 d\Omega_4^2 \right]. \quad (2.7)$$

where the function  $f_T(U)$  is given as<sup>5</sup>,

$$f_T(U) = 1 - \frac{U_T^3}{U^3}. \quad (2.8)$$

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<sup>5</sup>In the black D0-D4 solution, we replace  $U_{KK}$  by  $U_T$  in the solution.

The solution for the other fields in the black D0-D4 background could also be obtained after interchanging  $X^4$  and  $X^0$  in (2.2). The metric (2.7) describes a horizon at  $U = U_T$ , thus it corresponds to a quantum field theory at finite temperature.

As the Witten-Sakai-Sugimoto model, the flavors could be introduced by embedding a stack of  $N_f$  D8 and anti-D8 branes (D8/ $\overline{\text{D8}}$ -branes) as probes into the D0-D4 background (2.1) or (2.7). These D8/ $\overline{\text{D8}}$ -branes provide  $U_R(N_f) \times U_L(N_f)$  symmetry as chiral symmetry holographically. It has been turned out that, in the bubble solution (2.1), the D8/ $\overline{\text{D8}}$ -branes are always connected which represents the chirally broken symmetry in the dual field theory. In the black brane solution (2.7), the configuration of D8/ $\overline{\text{D8}}$ -branes could be connected or parallel, which could be identified as the chirally broken or symmetric phase in the dual field theory respectively<sup>6</sup>. The various configurations of D8/ $\overline{\text{D8}}$ -branes are shown in Figure 1 and Figure 2.

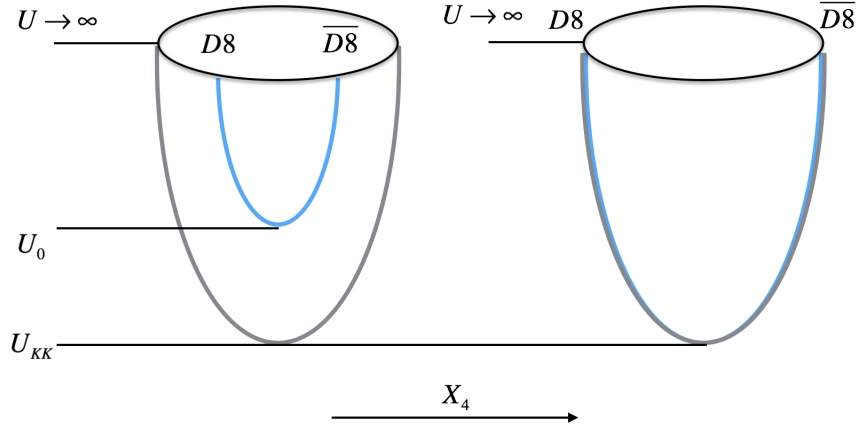


Figure 1: Possible configurations of D8/ $\overline{\text{D8}}$ -branes in the D0-D4 bubble background in  $X_4 - U$  plane. The D8/ $\overline{\text{D8}}$ -branes are always connected which represents the chirally broken phase in the dual field theory. The left configuration is non-antipodal while the right one is antipodal.

<sup>6</sup>The analyzing of the configuration of the D8/ $\overline{\text{D8}}$ -branes is similar as [24, 25] in the Witten-Sakai-Sugimoto model.

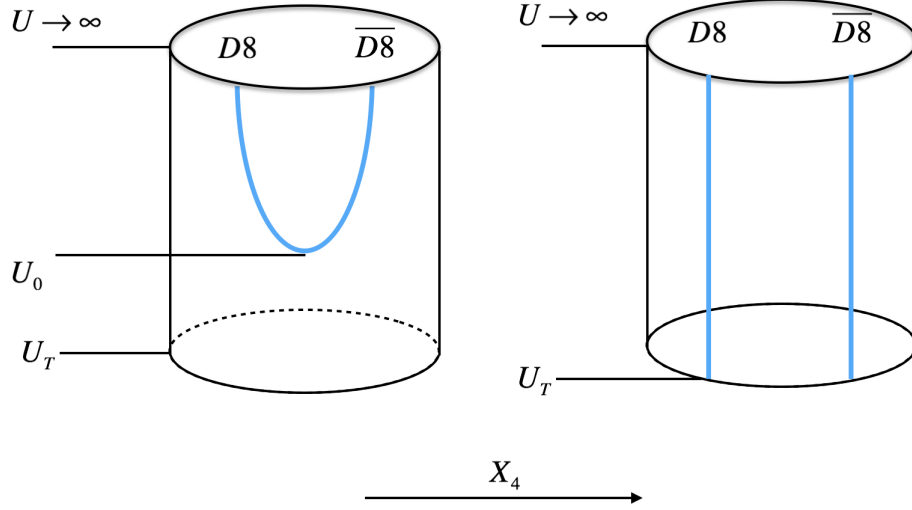


Figure 2: Possible configurations of  $D8/\overline{D8}$ -branes in the black D0-D4 background in  $X_4 - U$  plane. In the left one, the  $D8/\overline{D8}$ -branes are connected while they are parallel in the right one, which corresponds to chirally broken and symmetric phase in the dual field theory respectively.

### 3 Euler-Heisenberg Lagrangian of the D0-D4/D8 brane system

Although evaluating the Schwinger effect in strongly coupled gauge theories is challenging, the AdS/CFT correspondence becomes a powerful tool for analyzing such theories. By holography, it is allowed us to consider a classical gravity which is dual to the strongly coupled gauge theory. In this section, we are going to derive the Euler-Heisenberg Lagrangian in bubble and black D0-D4 brane background respectively. Then we can investigate the Schwinger effect or creation of quark-antiquark.

#### 3.1 Bubble D0-D4 geometry

In the bubble D0-D4 background, the flavored  $D8/\overline{D8}$ -branes are embedded with the following induced metric on their worldvolume,

$$ds_{D8}^2 = H_0^{1/2} \left( \frac{U}{R} \right)^{3/2} \eta_{\mu\nu} dX^\mu dX^\nu + H_0^{1/2} \left( \frac{U}{R} \right)^{3/2} \frac{dU^2}{h_c(U)} + H_0^{1/2} \left( \frac{R}{U} \right)^{3/2} U^2 d\Omega_4^2, \quad (3.1)$$

where

$$h_c(U) \equiv \left[ H_0^{-1} f(U) \left( \frac{dX^4(U)}{dU} \right)^2 + \left( \frac{R}{U} \right)^3 \frac{1}{f(U)} \right]^{-1}. \quad (3.2)$$

Notice that, for antipodal case, the D8-branes intersect  $X^4 = 0$  and the anti-D8-branes put parallel at  $X^4 = \pi R_{S^1}$  which implies  $\partial_U X^4 = 0$  in (3.2). Here, the  $R_{S^1}$  represents the radius of  $S^1$ .

In order to evaluate the vacuum decay rate in strongly coupled gauge theory, we need to derive the Euler-Heisenberg Lagrangian and analyze its instability (i.e. find the imaginary part of the Euler-Heisenberg Lagrangian). Since the Euler-Heisenberg Lagrangian could be identified as the probe brane action [12, 13, 14], in a word, we shall derive the Euler-Heisenberg Lagrangian from the following probe D8/ $\overline{\text{D8}}$ -brane action with a constant electromagnetic field<sup>7</sup>,

$$S_{\text{D8}/\overline{\text{D8}}}^{\text{DBI}} = -T_8 \int_{U_0}^{\infty} d^4 X dU d\Omega_4 e^{-\phi} \sqrt{-\det(P[g]_{ab} + 2\pi\alpha' F_{ab})}, \quad (3.3)$$

where  $T_8$  is the D8-brane tension which is defined as  $T_8 = g_s^{-1}(2\pi)^{-8} l_s^{-9}$  and  $U_0$  represents the connected position of the D8/ $\overline{\text{D8}}$ -branes<sup>8</sup>. For simplicity, we need to consider a single flavor  $N_f = 1$  for the presence of an external electromagnetic field. We further require the electromagnetic field is non-dynamical (i.e. constant) and their components on the  $S^4$  are zero. Without losing generality, we could turn on the electric field on the  $X^1$  direction only and the magnetic fields could be introduced in  $X^1, X^2, X^3$  directions since the  $X^i$ ,  $i = 1, 2, 3$  spacial space is rotationally symmetric. Inserting constant electromagnetic field with the induced metric (3.1) into the DBI action (3.3), the effective Lagrangian is obtained as,

$$\mathcal{L} = -\frac{8\pi^2 T_8}{3g_s} \int_{U_0}^{\infty} dU U^4 H_0^{3/2} h_c^{-1/2} \sqrt{\xi_c}, \quad (3.4)$$

where the integral of  $d\Omega_4$  is  $\text{Vol}(S^4) = 8\pi^2/3$ . And  $\xi_c$  is given as,

$$\begin{aligned} \xi_c = & 1 - \frac{(2\pi\alpha')^2 R^3}{U^3 H_0} [F_{01}^2 - F_{12}^2 - F_{23}^2 - F_{13}^2 + h_c(U) (F_{0U}^2 - F_{1U}^2)] \\ & - \frac{(2\pi\alpha')^4 R^6}{U^6 H_0^2} [F_{01}^2 F_{23}^2 + h_c(U) \{F_{0U}^2 (F_{12}^2 + F_{23}^2 + F_{13}^2) - F_{1U}^2 F_{23}^2\}]. \end{aligned} \quad (3.5)$$

Then, we will derive the equations of motion from (3.4) respected to  $A_0(U)$  and  $A_1(U)$ . Notice that, we could put  $\partial_i = 0$ ,  $i = 1, 2, 3$  since only the homogeneous phases are interesting here. In particular, we additionally require  $\partial_0 = 0$  as a constraint of the static configurations, so that the equations of motion for the static (time-independent) configurations are obtained as<sup>9</sup>,

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<sup>7</sup>The low energy effective action of a D-brane consists of two parts: Dirac-Born-Infeld (DBI) action plus Chern-Simons (CS) term. In this paper, we do not need to consider the Chern-Simons term since it has nothing to do with the electromagnetic instability.

<sup>8</sup>For the antipodal case, we need to choose  $U_0 = U_{KK}$ .

<sup>9</sup>When both electric and magnetic fields are turned on, the Chern-Simons term in the action contributes to the equations of motion. It turns out that the equations of motion for  $A_U$  implies the chiral anomaly. For simplicity, we can ignore this anomaly effect by interpreting our outcome as the physical values measured at  $t = 0$ , at which  $A_U$  vanishes as an initial condition.



$$\begin{aligned}\partial_U \left[ \frac{UH_0^{1/2}h_c^{1/2}F_{0U}}{\sqrt{\xi_c}} \left\{ 1 + \frac{(2\pi\alpha')^2 R^3}{U^3 H_0} (F_{12}^2 + F_{13}^2 + F_{23}^2) \right\} \right] &= 0, \\ \partial_U \left[ \frac{UH_0^{1/2}h_c^{1/2}F_{1U}}{\sqrt{\xi_c}} \left\{ 1 + \frac{(2\pi\alpha')^2 R^3 F_{23}^2}{U^3 H_0} \right\} \right] &= 0.\end{aligned}\quad (3.6)$$

According to the equations of motion, the definition of the number density  $d$  and the current  $j$  reads,

$$\begin{aligned}d &= \frac{(2\pi\alpha')^2 8\pi^2 R^3 T_8}{3g_s} \frac{UH_0^{1/2}h_c^{1/2}F_{0U}}{\sqrt{\xi_c}} \left\{ 1 + \frac{(2\pi\alpha')^2 R^3}{U^3 H_0} (F_{12}^2 + F_{13}^2 + F_{23}^2) \right\}, \\ j &= \frac{(2\pi\alpha')^2 8\pi^2 R^3 T_8}{3g_s} \frac{UH_0^{1/2}h_c^{1/2}F_{1U}}{\sqrt{\xi_c}} \left\{ 1 + \frac{(2\pi\alpha')^2 R^3}{U^3 H_0} F_{23}^2 \right\}.\end{aligned}\quad (3.7)$$

Substituting (3.4) for (3.7), we have

$$\begin{aligned}\xi_c &= \frac{1 - \frac{(2\pi\alpha')^2 R^3}{U^3 H_0} (F_{01}^2 - F_{12}^2 - F_{13}^2 - F_{23}^2) - \frac{(2\pi\alpha')^4 R^6}{U^6 H_0^2} F_{01}^2 F_{23}^2}{1 + \frac{9g_s^2}{2^6 \pi^4 (2\pi\alpha')^2 R^3 T_8^2 U^5 H_0^2} \left[ \frac{d^2}{1 + \frac{(2\pi\alpha')^2 R^3}{U^3 H_0} (F_{12}^2 + F_{13}^2 + F_{23}^2)} - \frac{j^2}{1 + \frac{(2\pi\alpha')^2 R^3}{U^3 H_0} F_{23}^2} \right]}.\end{aligned}\quad (3.8)$$

Therefore the effective Lagrangian as the Euler-Heisenberg Lagrangian at zero temperature is,

$$\begin{aligned}\mathcal{L} &= -\frac{8\pi^2 T_8}{3g_s} \int_{U_{KK}}^\infty dU U^4 H_0^{3/2} h_c^{-1/2} \\ &\times \sqrt{\frac{1 - \frac{(2\pi\alpha')^2 R^3}{U^3 H_0} (E_1^2 - \vec{B}^2) - \frac{(2\pi\alpha')^4 R^6}{U^6 H_0^2} E_1^2 B_1^2}{1 + \frac{9g_s^2}{2^6 \pi^4 (2\pi\alpha')^2 R^3 T_8^2 U^5 H_0^2} \left[ \frac{d^2}{1 + \frac{(2\pi\alpha')^2 R^3}{U^3 H_0} \vec{B}^2} - \frac{j^2}{1 + \frac{(2\pi\alpha')^2 R^3}{U^3 H_0} B_1^2} \right]}},\end{aligned}\quad (3.9)$$

where we have defined the constant electric and magnetic field as  $F_{0i} = E_i$ ,  $\epsilon_{ijk} F_{jk} = B_i$  and  $\vec{B}^2 = B_1^2 + B_2^2 + B_3^2$ .

### 3.2 Black D0-D4 geometry

In the black D0-D4 background, there are two possible configurations for the embedded D8/ $\overline{\text{D8}}$ -branes which are connected and parallel respectively as shown in Figure 2. Generically, the induced metric on the D8/ $\overline{\text{D8}}$ -branes could be written as,

$$\begin{aligned}
ds^2 = & \left(\frac{U}{R}\right)^{3/2} \left[ -H_0^{-1/2} f_T(U) dt^2 + H_0^{1/2} \delta_{ij} dX^i dX^j \right] \\
& + \left(\frac{U}{R}\right)^{3/2} H_0^{1/2} \frac{dU^2}{h_d(U)} + H_0^{1/2} \left(\frac{R}{U}\right)^{3/2} U^2 d\Omega_4^2,
\end{aligned} \tag{3.10}$$

where

$$h_d(U) = \left[ (\partial_U X^4)^2 + \left(\frac{R}{U}\right)^3 \frac{1}{f_T(U)} \right]^{-1}. \tag{3.11}$$

For connected configuration of the D8/ $\overline{\text{D8}}$ -branes, we keep  $X^4$  in (3.11) as a generic function depended on  $U$ , while  $X^4$  is a constant for parallel configuration. Then similar as done in the bubble case, we could obtain the following Euler-Heisenberg Lagrangian once the induced metric (3.10) and the constant electromagnetic fields are adopted,

$$\mathcal{L} = -\frac{8\pi^2 T_8}{3g_s} \int_{U_0, U_T}^{\infty} dU U^4 H_0 h_d^{-1/2} f_T^{1/2} \sqrt{\xi_d}. \tag{3.12}$$

where

$$\begin{aligned}
\xi_d = & 1 - \frac{(2\pi\alpha')^2 R^3}{U^3 H_0 f_T} [H_0 (F_{01}^2 + F_{0U}^2 h_d) - f_T (F_{12}^2 + F_{13}^2 + F_{23}^2 + F_{1U}^2 h_d)] \\
& - \frac{(2\pi\alpha')^4 R^6}{U^6 H_0^2 f_T} [H_0 F_{01}^2 F_{23}^2 + H_0 h_d F_{0U}^2 (F_{12}^2 + F_{13}^2 + F_{23}^2) - h_d f_T F_{1U}^2 F_{23}^2].
\end{aligned} \tag{3.13}$$

We have used  $U_0$  to represent the connected position of the D8/ $\overline{\text{D8}}$ -branes. Notice that in the parallel configuration, the integral in (3.12) starts from  $U_T$ . Next, we could obtain the equations of motion for static (time independent) configuration which are,

$$\begin{aligned}
\partial_U \left[ \frac{U H_0 h_d^{1/2} F_{0U}}{\sqrt{f_T \xi_d}} \left\{ 1 + \frac{(2\pi\alpha')^2 R^3}{U^3 H_0} (F_{12}^2 + F_{13}^2 + F_{23}^2) \right\} \right] &= 0, \\
\partial_U \left[ \frac{U h_d^{1/2} f_T^{1/2} F_{1U}}{\sqrt{\xi_d}} \left( 1 + \frac{(2\pi\alpha')^2 R^3}{U^3 H_0} F_{23}^2 \right) \right] &= 0.
\end{aligned} \tag{3.14}$$

So we have the charge density  $d$  and current  $j$  defined as,

$$\begin{aligned}
d &= \frac{(2\pi\alpha')^2 8\pi^2 R^3 T_8}{3g_s} \frac{U H_0 h_d^{1/2} F_{0U}}{\sqrt{f_T \xi_d}} \left\{ 1 + \frac{(2\pi\alpha')^2 R^3}{U^3 H_0} (F_{12}^2 + F_{13}^2 + F_{23}^2) \right\}, \\
j &= \frac{(2\pi\alpha')^2 8\pi^2 R^3 T_8}{3g_s} \frac{U h_d^{1/2} f_T^{1/2} F_{1U}}{\sqrt{\xi_d}} \left[ 1 + \frac{(2\pi\alpha')^2 R^3}{U^3 H_0} F_{23}^2 \right].
\end{aligned} \tag{3.15}$$

Therefore the onshell effective Lagrangian (3.12) contains the following form,

$$\xi_d = \frac{1 - \frac{(2\pi\alpha')^2 R^3}{U^3 H_0 f_T} [H_0 F_{01}^2 - f_T (F_{12}^2 + F_{13}^2 + F_{23}^2)] - \frac{(2\pi\alpha')^4 R^6}{U^6 H_0 f_T} F_{01}^2 F_{23}^2}{1 + \frac{3^2 g_s^2}{2^6 \pi^4 (2\pi\alpha')^2 R^3 T_8^2 U^5 H_0^2 f_T} \left[ \frac{d^2 f_T}{1 + \frac{(2\pi\alpha')^2 R^3}{U^3 H_0} (F_{12}^2 + F_{13}^2 + F_{23}^2)} - \frac{j^2 H_0}{1 + \frac{(2\pi\alpha')^2 R^3}{U^3 H_0} F_{23}^2} \right]}. \quad (3.16)$$

Accordingly, the Euler-Heisenberg Lagrangian at finite temperature is,

$$\mathcal{L} = - \frac{8\pi^2 T_8}{3g_s} \int_{U_0, U_T}^{\infty} dU U^4 H_0 h_d^{-1/2} f_T^{1/2} \times \sqrt{\frac{1 - \frac{(2\pi\alpha')^2 R^3}{U^3 H_0 f_T} [H_0 E_1^2 - f_T \vec{B}^2] - \frac{(2\pi\alpha')^4 R^6}{U^6 H_0 f_T} E_1^2 B_1^2}{1 + \frac{3^2 g_s^2}{2^6 \pi^4 (2\pi\alpha')^2 R^3 T_8^2 U^5 H_0^2 f_T} \left[ \frac{d^2 f_T}{1 + \frac{(2\pi\alpha')^2 R^3}{U^3 H_0} \vec{B}^2} - \frac{j^2 H_0}{1 + \frac{(2\pi\alpha')^2 R^3}{U^3 H_0} B_1^2} \right]}} \quad (3.17)$$

where the definitions of  $E_i$  and  $B_i$  are as same as in the bubble case.

## 4 Holographic pair creation of quark-antiquark

In the previous sections, we have obtained the effective Euler-Heisenberg Lagrangian both in the bubble and black D0-D4 brane geometry. In this section, let us compute the imaginary part of the action in order to evaluate the pair creation of quark-antiquark holographically at zero or finite temperature. For simplicity, we will be interested in looking at the instability of the vacuum as [12, 13, 14].

### 4.1 Imaginary part of the effective action at zero temperature

Since the bubble D0-D4 geometry holographically corresponds to the confinement phase of the dual field theory at zero temperature, let us evaluate the imaginary part of the action (3.9) from the bubble D0-D4 geometry first. For the vacuum case, by setting  $d = j = 0$ , the Lagrangian (3.9) takes the following form,

$$\mathcal{L} = - \frac{8\pi^2 T_8}{3g_s} \int_{U_0, U_{KK}}^{\infty} dU U^4 H_0^{3/2} h_c^{-1/2} \sqrt{1 - \frac{(2\pi\alpha')^2 R^3}{U^3 H_0} (E_1^2 - \vec{B}^2) - \frac{(2\pi\alpha')^4 R^6}{U^6 H_0^2} E_1^2 B_1^2}. \quad (4.1)$$

The critical position for  $U = U_*$ , where the imaginary part of the Lagrangian (4.1) appears, obviously satisfies the equation,

$$\frac{(2\pi\alpha')^4 R^6}{U_*^6 H_0 (U_*)^2} E_1^2 B_1^2 + \frac{(2\pi\alpha')^2 R^3}{U_*^3 H_0 (U_*)} (E_1^2 - \vec{B}^2) - 1 = 0. \quad (4.2)$$

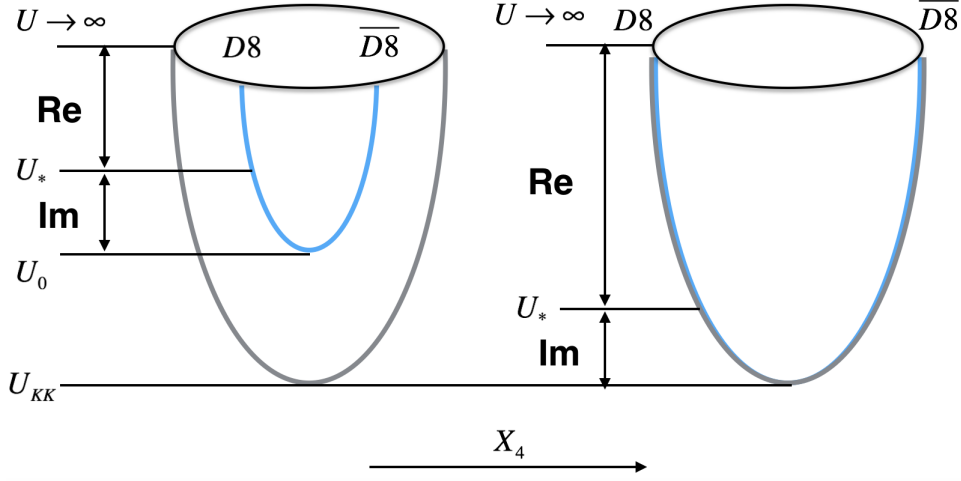


Figure 3: The region of the integral in the imaginary Lagrangian of the D8/ $\overline{D8}$ -branes in the bubble D0-D4 background. **Left:** In the non-antipodal case, the integral in the imaginary Lagrangian starts from  $U_0$  to  $U_*$ . **Right:** In the antipodal case the integral in the imaginary Lagrangian starts from  $U_{KK}$  to  $U_*$ .

After solving (4.2) we therefore obtain,

$$U_* = \left\{ \frac{(2\pi\alpha')^2 R^3}{2} \left[ E_1^2 - \vec{B}^2 + \sqrt{(\vec{B}^2 - E_1^2)^2 + 4B_1^2 E_1^2} \right] - U_{Q_0}^3 \right\}^{1/3}. \quad (4.3)$$

So it is clear that in the region of  $U : U_0 \leq U \leq U_*$ , the Lagrangian is imaginary as shown in Figure 3.

Since the creation rate of the quark-antiquark is proportional to the imaginary part of the Lagrangian (4.1), next we are going to examine whether or not the imaginary part diverges. By the neighborhood of  $U_0$ , we assume  $U_* = U_0 + \varepsilon$  where  $\varepsilon \ll U_0$  as [12]. Then the imaginary part of (4.1) can be rewritten in terms of the expansion near  $U_0$  as,

$$\begin{aligned} \text{Im}\mathcal{L} &= -\frac{8\pi^2 T_8}{3g_s} \int_{U_0, U_{KK}}^{U_*} dU U^4 H_0^{3/2} h_c^{-1/2} \sqrt{\frac{(2\pi\alpha')^4 R^6}{U^6 H_0^2} E_1^2 B_1^2 + \frac{(2\pi\alpha')^2 R^3}{U^3 H_0} (E_1^2 - \vec{B}^2)} - 1 \\ &\simeq -\frac{8\pi^2 T_8}{3g_s} \mathcal{F}(U_0) \lim_{U_* \rightarrow U_0} \left[ (U_* - U_0) \times h_c^{-1/2}(U_0) \right] \\ &\simeq -\frac{8\pi^2 T_8}{3g_s} \mathcal{F}(U_0) \left[ H_0^{-1}(U_0) f(U_0) [X^4(U_0 + \varepsilon) - X^4(U_0)]^2 + \frac{R^3 \varepsilon^2}{(U_0 + \varepsilon)^3 - U_{KK}^3} \right]^{1/2}, \end{aligned} \quad (4.4)$$

where  $\mathcal{F}(U)$  is defined as,

$$\mathcal{F}(U) = U^4 H_0^{3/2} \sqrt{\frac{(2\pi\alpha')^4 R^6}{U^6 H_0^2} E_1^2 B_1^2 + \frac{(2\pi\alpha')^2 R^3}{U^3 H_0} (E_1^2 - \vec{B}^2)} - 1. \quad (4.5)$$

Consequently (4.4) is finite for the non-antipodal case (i.e.  $U_0 > U_{KK}$ ). Moreover, in the antipodal case, we must require  $U_0 = U_{KK}$ . So  $X^4(U_0 + \varepsilon) - X^4(U_0)$  vanishes since  $X^4$  is a constant. Thus (4.4) could be simplified as,

$$\begin{aligned} \text{Im}\mathcal{L} &\simeq \frac{8\pi^2 T_8 R^{3/2}}{3g_s} \mathcal{F}(U_{KK}) \frac{\varepsilon}{\sqrt{(U_{KK} + \varepsilon)^3 - U_{KK}^3}} \\ &\simeq \frac{8\pi^2 T_8 R^{3/2}}{3g_s U_{KK}} \mathcal{F}(U_{KK}) \left[ \sqrt{\frac{\varepsilon}{3}} + \mathcal{O}(\varepsilon^{3/2}) \right] = \text{finite}. \end{aligned} \quad (4.6)$$

So (4.4) shows a finite value for the creation rate of quark-antiquark in our D0-D4/D8 system which is due to the confining scale  $U_{KK}$ . If setting  $U_{Q_0} = 0$  (i.e. no smeared D0-branes or vanished  $\theta$  angle), our result returns to the approximated approach in the Witten-Sakai-Sugimoto model as [12]. Accordingly, it is natural to treat (4.4) as a holographically generic form of the creation rate of quark-antiquark which is dependent on the topological charge from the QCD vacuum.

Since the creation of the quark antiquark breaks the vacuum, we need to evaluate the critical electric field. By the condition that the Lagrangian (4.4) begins to be imaginary, the critical electric field could be derived by solving,

$$U_0 = U_* = \left\{ \frac{(2\pi\alpha')^2 R^3}{2} \left[ E_1^2 - \vec{B}^2 + \sqrt{(\vec{B}^2 - E_1^2)^2 + 4B_1^2 E_1^2} \right] - U_{Q_0}^3 \right\}^{1/3}. \quad (4.7)$$

Thus the critical electric field is,

$$E_{cr} = \left[ \frac{U_0^3 + U_{Q_0}^3}{(2\pi\alpha')^2 R^3} \frac{\left\{ \frac{U_0^3 + U_{Q_0}^3}{(2\pi\alpha')^2 R^3} + \vec{B}^2 \right\}}{\left\{ \frac{U_0^3 + U_{Q_0}^3}{(2\pi\alpha')^2 R^3} + B_1^2 \right\}} \right]^{1/2}. \quad (4.8)$$

We find the critical electric field does not depend on  $B_1$  if setting  $B_2, B_3 = 0$ . Besides, if we look at the antipodal case which means  $U_0 = U_{KK}$ , and substitute (4.8) for (2.6), then the critical electric field could be obtained as,

$$E_{cr} = \frac{2}{27\pi} \lambda M_{KK}^2 (1 + \zeta)^2 \left[ \frac{\frac{4}{3^6 \pi^2} \lambda^2 M_{KK}^4 (1 + \zeta)^4 + \vec{B}^2}{\frac{4}{3^6 \pi^2} \lambda^2 M_{KK}^4 (1 + \zeta)^4 + B_1^2} \right]^{1/2}. \quad (4.9)$$

where  $\zeta$  is defined as  $\zeta = U_{Q_0}^3 / U_{KK}^3$ . Since  $\zeta$  is related to the D0-brane density, (4.9) shows the dependence on the  $\theta$  angle from the QCD vacuum. And it coincides with the generic

formula in [12] if setting  $\zeta = 0$ . Notice that if  $B_2 = B_3 = 0$ , the critical electric field is  $E_{cr} = \frac{2}{27\pi} \lambda M_{KK}^2 (1 + \zeta)^2$  which increases by the appearance of the  $\theta$  angle (i.e. D0 charge).

In order to evaluate the imaginary part of the Lagrangian (4.1) numerically, let us derive the expression of (4.1) by using the following dimensionless variables. Introducing the dimensionless variables as,

$$U = \frac{U_{KK}}{y H_{KK} H_0^{1/3}} = \frac{\left( U_{KK}^3 - H_{KK}^3 U_{Q_0}^3 y^3 \right)^{1/3}}{y H_{KK}} \quad (4.10)$$

and substituting (4.1) for (2.6) and (4.10), we obtain,

$$\begin{aligned} \text{Im}\mathcal{L} = & \frac{M_{KK}^4 \lambda^3 N_c}{2 \cdot 3^8 \pi^5} \int_{y_*}^{y_0, y_{kk}} dy \left\{ 1 + y \left[ 1 - y^3 \zeta (1 + \zeta)^3 \right]^{4/3} \left[ 1 - y^3 (1 + \zeta)^4 \right]^2 x_4'^2 \right\}^{1/2} \\ & \times y^{-9/2} \left[ 1 - y^3 \zeta (1 + \zeta)^3 \right]^{-5/6} \left[ 1 - y^3 (1 + \zeta)^4 \right]^{-1/2} \\ & \times \sqrt{y^6 \mathbf{E}_1^2 \mathbf{B}_1^2 + y^3 \left( \mathbf{E}_1^2 - \mathbf{B}^2 \right) - 1}, \end{aligned} \quad (4.11)$$

where

$$\begin{aligned} y_{kk} &= (1 + \zeta)^{-4/3}, \quad y_* = \frac{U_{KK}}{(1 + \zeta) \left( U_*^3 + U_{Q_0}^3 \right)^{1/3}}, \\ X_4 &= \frac{3x_4}{2M_{KK}}, \quad \mathbf{E}_i = \frac{3^3 \pi}{2\lambda M_{KK}^2} E_i, \quad \mathbf{B}_i = \frac{3^3 \pi}{2\lambda M_{KK}^2} B_i. \end{aligned} \quad (4.12)$$

Notice that we need to obtain the exact formula for  $x_4$  in (4.11) by solving its equation of motion before the numerical calculations, however, which would become very challenging. So in this paper, we will not attempt to evaluate (4.11) with the exact solution for  $x_4$ . Instead, as a typical exploration, we evaluate the imaginary part (4.1) in the antipodal case for simplicity<sup>10</sup>. Moreover, we interestingly find that (4.11) may show an additionally possible instability because the second line of (4.11) could also be imaginary and it does not depend on the electromagnetic field. Since the bubble D0-D4/D8 system corresponds to a confining Yang-Mills theory with a topological Chern-Simons term, so the instability produced by  $\theta$  angle (i.e. D0 charge) could be holographically interpreted as the transition between the different  $\theta$  vacuum states, which has been very well-known in QCD. However in order to investigate the electromagnetic instability, we need to remove the  $\theta$ -instability from the vacuum. Hence we expand (4.11) by  $\zeta$  since  $\theta$  angle

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<sup>10</sup>In some limit, the contribution from  $x_4'$  to the effective Lagrangian is not important. For example, if  $\varepsilon \rightarrow 0$  in (4.4), we obtain  $X^4(U_0 + \varepsilon) \simeq X^4(U_0)$ . Therefore in this limit,  $x_4'$  does not contribute to the effective Lagrangian (4.4).

in QCD is very small. Therefore in the antipodal case (i.e.  $x'_4 = 0$ ) with small  $\zeta$  expansion, we obtain the following formula for the imaginary Lagrangian,

$$\begin{aligned} \text{Im}\mathcal{L} \simeq & \frac{M_{KK}^4 \lambda^3 N_c}{2 \cdot 3^8 \pi^5} \int_{y_*}^{y_{kk}} dy \sqrt{y^6 \mathbf{E}_1^2 \mathbf{B}_1^2 + y^3 (\mathbf{E}_1^2 - \vec{\mathbf{B}}^2)} - 1 \\ & \times y^{-9/2} \left[ \frac{1}{\sqrt{1-y^3}} - \frac{y^3 (5y^3 - 17)}{6 (1-y^3)^{3/2}} \zeta \right]. \end{aligned} \quad (4.13)$$

The (4.13) could be numerically evaluated and the result is shown in Figure 4. We plot the  $\text{Im}L$  as a function of  $\zeta$ , the dimensionless magnetic field  $B_P$  and  $B_V$  which is parallel and perpendicular to the (dimensionless) electric field  $\mathbf{E}$ . In Figure 4, we use different colors, as red, blue and green, to distinguish the dependence on  $\zeta$  ( $\theta$  angle) with  $\zeta = 0.5, 0.3, 0$  respectively. It shows if the magnetic field and the electric field are parallel, the imaginary part of the Lagrangian increases as the magnetic field increases. And on the other hand, if the magnetic field and the electric field is perpendicular to each other, the imaginary part of the Lagrangian decreases when the magnetic field increases. The relation between  $\text{Im}L$  and parallel/perpendicular magnetic field with different  $\zeta$  is shown in Figure 5.

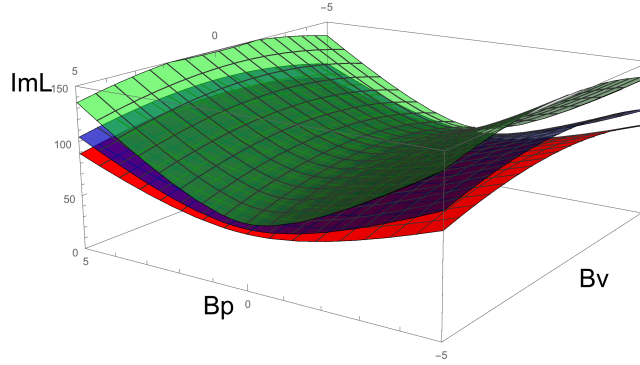


Figure 4: The imaginary part of the effective Lagrangian with a fixed  $E$  as a function of  $\zeta$ , the magnetic field  $B_P$  and  $B_V$  which is parallel and perpendicular to the electric field  $E$ . In this figure, we look at  $\mathbf{E}_1 = E = 10$  with  $\zeta = 0.5, 0.3, 0$  represented by red, blue and green respectively. It shows the dependence on  $\zeta$  ( $\theta$  angle) and the imaginary part of the Lagrangian decreases if  $\zeta$  increases.

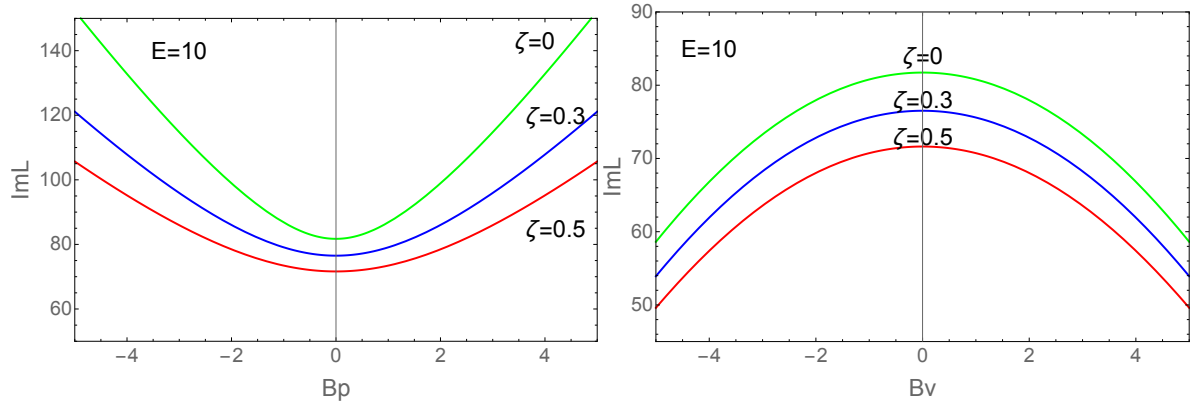


Figure 5: Relation between  $\text{Im}L$  and  $B_V$ ,  $B_P$  with different  $\zeta$ . If  $\zeta$  increases,  $\text{Im}L$  decreases. **Left:** The magnetic field is parallel to the electric field and  $\text{Im}L$  increases as the magnetic field increases. **Right:** The magnetic field is perpendicular to the electric field and  $\text{Im}L$  decreases as the magnetic field increases.

We furthermore look at the dependence on  $E$  with various  $\zeta$  in the case of a parallel/perpendicular magnetic field respectively. The numerical evaluation is summarized in Figure 6. Accordingly, we could conclude that, in the bubble D0-D4 system, the electromagnetic instability is suppressed by the appearance of D0 charge ( $\theta$  angle in QCD), however with a fixed  $\zeta$ , its behavior depends on the direction of the magnetic field relative to the electric field. Finally, we also plot the relation between the electric field with an arbitrary magnetic field to confirm our conclusion which is shown in Figure 7.

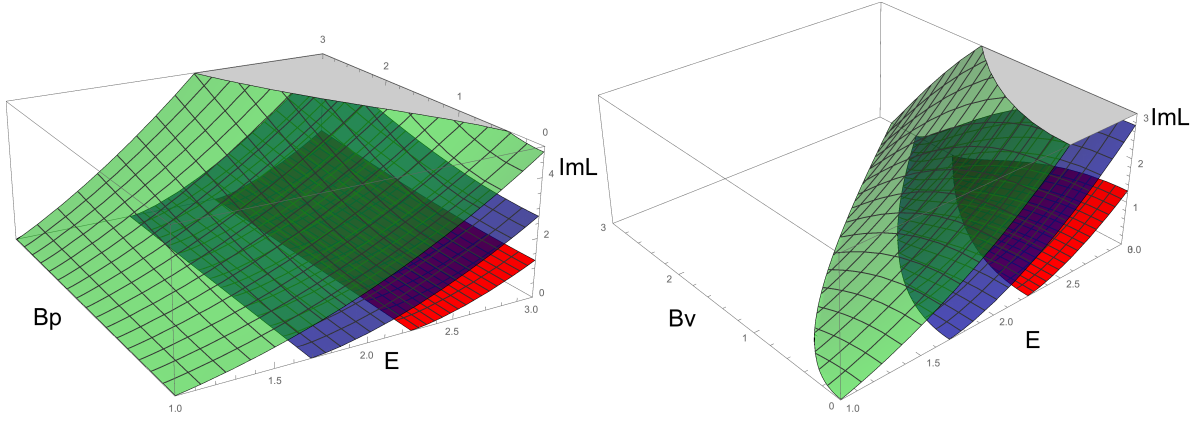


Figure 6: In this figure, red blue and green represent  $\zeta = 0.5$ ,  $0.3$ ,  $0$  respectively as before. **Left:** The magnetic field is parallel to the electric field. **Right:** The magnetic field is perpendicular to the electric field.



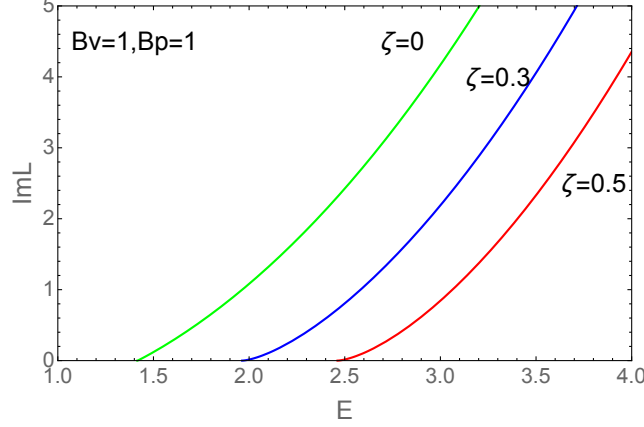


Figure 7: The relation between  $\text{Im}L$  with an arbitrary magnetic field and various  $\zeta$ .

## 4.2 Imaginary part of the action at finite temperature

In the previous section, we have obtained the effective action from the black D0-D4 background i.e. at finite temperature. Since we are most interested in the vacuum instability, it would be suitable to set the charge density and current vanished in the Lagrangian (3.17) i.e.  $d = j = 0$ . Then we obtain the simplified Lagrangian from (3.17) which is,

$$\mathcal{L} = -\frac{8\pi^2 T_8}{3g_s} \int_{U_0, U_T}^{\infty} dU U^4 H_0 h_d^{-1/2} f_T^{1/2} \times \sqrt{1 - \frac{(2\pi\alpha')^2 R^3}{U^3 H_0 f_T} [H_0 E_1^2 - f_T \vec{B}^2]} - \frac{(2\pi\alpha')^4 R^6}{U^6 H_0 f_T} E_1^2 B_1^2. \quad (4.14)$$

We can evaluate the imaginary part of the Lagrangian to derive the rate of the quark antiquark creation in the vacuum by (4.14).

To begin with, let us consider the case of zero-temperature limit, i.e.  $U_T \rightarrow 0$ , so that the function  $f_T(U)$  approaches unity. The third term in the square root of the Lagrangian (4.14) must be dominated because the integral in (4.14) should be totally imaginary. Thus the  $U$ -integral is finite. Accordingly, in the presence of the electromagnetic field, the vacuum decay rate is finite at strong coupling in the zero temperature limit in our D0-D4/D8 system or the Witten-Sakai-Sugimoto model. Then let us compute it in details. Introducing the dimensionless variables as,

$$y = \frac{U_T}{U H_T H_0^{1/3}}, \quad \zeta_T = \frac{U_{Q_0}^3}{U_T^3}, \quad \chi = \frac{U_T^3}{(2\pi\alpha')^2 R^3}, \quad x_4 = \frac{3}{2} \frac{3U_T^{1/2}}{2R^{3/2}} H_T^{-1/2} X^4, \quad (4.15)$$

we have,

$$\begin{aligned}
\mathcal{L} = & -\frac{2^3\pi^2(2\pi\alpha')^{7/3}R^5T_8}{3g_s}\chi^{7/6}(1+\zeta_T)^{-1/2}\int_{y_T,y_0}^0 dy y^{-9/2}\left[1-y^3\zeta_T(1+\zeta_T)^3\right]^{-5/6} \\
& \times \left\{y\left[1-y^3(1+\zeta_T)^4\right]\left[1-y^3\zeta_T(1+\zeta_T)^3\right]^{4/3}x_4'^2+1\right\}^{1/2} \\
& \times \sqrt{(1+\zeta_T)^{-6}-\frac{y^3}{\chi}\left\{E_1^2\left[1-y^3(1+\zeta_T)^4\right]^{-1}-\vec{B}^2\right\}(1+\zeta_T)^{-3}-\frac{y^6}{\chi^2}E_1^2B_1^2\left[1-y^3(1+\zeta_T)^4\right]^{-1}},
\end{aligned} \tag{4.16}$$

where  $y_T = (1+\zeta_T)^{-4/3}$  and  $H_T \equiv H_0(U_T)$ . Further rescale  $Y = \chi^{-1/3}y$ , it yields,

$$\begin{aligned}
\text{Im}\mathcal{L} = & -\frac{2^3\pi^2(2\pi\alpha')^{7/3}R^5T_8}{3g_s}(1+\zeta_T)^{-7/2}\int_{\chi^{-1/3}(1+\zeta_T)^{-4/3}}^{Y_*} dY Y^{-9/2} \\
& \times \left\{\left[B_1^2E_1^2(1+\zeta_T)^6+\vec{B}^2(1+\zeta_T)^7\chi\right]Y^6\right. \\
& \left.+\left[(1+4\zeta_T+6\zeta_T^2+4\zeta_T^3+\zeta_T^4)\chi+\left(E_1^2-\vec{B}^2\right)(1+\zeta_T)^3\right]Y^3-1\right\}^{1/2} \\
& \times \left\{1+\chi^{1/3}Y\left[1-Y^3\zeta_T(1+\zeta_T)^3\chi\right]^{4/3}\left[1-Y^3(1+\zeta_T)^4\chi\right]x_4'^2\right\}^{1/2} \\
& \times \left[1-Y^3\zeta_T(1+\zeta_T)^3\chi\right]^{-5/6}\left[1-Y^3(1+\zeta_T)^4\chi\right]^{-1/2}.
\end{aligned} \tag{4.17}$$

Since there are two possible configurations for D8/ $\overline{\text{D8}}$ -branes in the black brane background as shown in Figure 8, let us consider the parallel D8/ $\overline{\text{D8}}$ -branes first, i.e.  $x_4' = 0$ . In the small temperature limit  $\chi \rightarrow 0$ , the third term in the square root of the Lagrangian (4.14) become dominated, so that we have the following behavior from (4.17),

$$\begin{aligned}
\text{Im}\mathcal{L} \simeq & -\frac{2^3\pi^2(2\pi\alpha')^{7/3}R^5T_8}{3g_s}\vec{E}\cdot\vec{B}(1+\zeta_T)^{-1/2}\int_{\chi^{-1/3}(1+\zeta_T)^{-4/3}}^{Y_*} dY Y^{-3/2} \\
& \times \left[1-Y^3\zeta_T(1+\zeta_T)^3\chi\right]^{-5/6}\left[1-Y^3\zeta_T(1+\zeta_T)^4\chi\right]^{-1/2} \\
& \simeq \frac{2^4\pi^2(2\pi\alpha')^{7/3}R^5T_8}{3g_s\sqrt{1+\zeta_T}}\vec{B}\cdot\vec{E}Y_*^{-1/2}+\mathcal{O}\left(\chi^{1/6}\right)=\text{finite}.
\end{aligned} \tag{4.18}$$

where  $Y_*$  satisfies the following equation in the small  $\chi$  limit,

$$1-\left(E_1^2-\vec{B}^2\right)Y_*^3-(B_1E_1)^2Y_*^6=0, \tag{4.19}$$

so that if the magnetic field is parallel to the electric field, we have  $Y_* = E_1^{-2/3}$ . According to (4.18), the creation rate decreases when  $\zeta_T$  increases. It coincides with our result in bubble case. Moreover, the generic solution of (4.19) can be found as,

$$Y_* = \left[ \frac{\vec{E}^2 - \vec{B}^2 + \sqrt{(\vec{E}^2 - \vec{B}^2)^2 + 4(\vec{E} \cdot \vec{B})^2}}{2} \right]^{-1/3}. \quad (4.20)$$

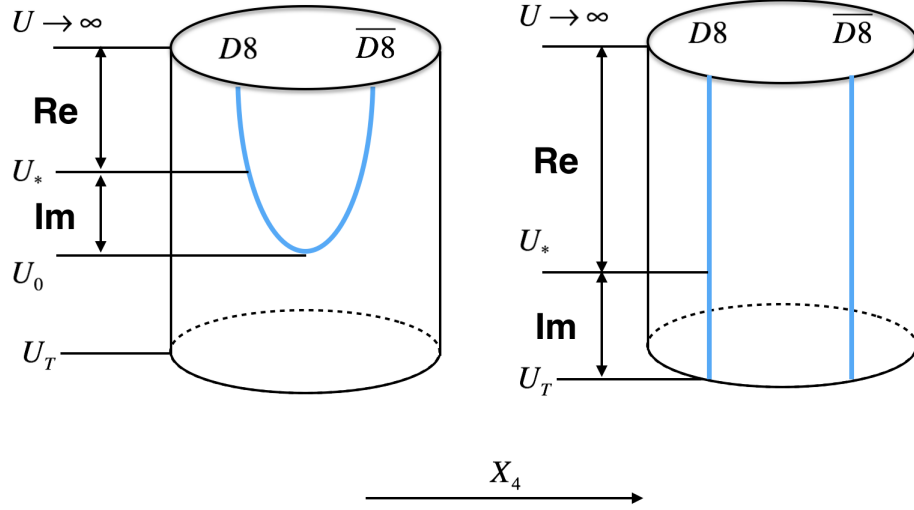


Figure 8: The region of the integral in the imaginary Lagrangian of the D8/ $\overline{D8}$ -branes in the black D0-D4 background. **Left:** The configuration of D8/ $\overline{D8}$ -branes is “U” shape, thus the integral in the imaginary Lagrangian starts from  $U_0$  to  $U_*$ . **Right:** The configuration of D8/ $\overline{D8}$ -branes is parallel and the integral in the imaginary Lagrangian starts from  $U_T$  to  $U_*$ .

On the other hand, for the connected configuration of D8/ $\overline{D8}$ -branes, taking the zero temperature limit  $U_T \rightarrow 0$ , we have,

$$y_0 = \frac{U_T}{U_0 H_T H_0^{1/3}(U_0)} \rightarrow 0. \quad (4.21)$$

Thus the imaginary part of the Lagrangian contains the integral starting from  $Y_0$  to  $Y_*$  (i.e.  $U_0$  to  $U_*$ ) with  $Y_0 \rightarrow Y_*$ . As the previous section, we can expand  $Y$  (or  $y$ ) in the neighborhood  $Y_* \simeq Y_0 + \varepsilon$  where  $\varepsilon \ll Y_0$ . Since in the connected configuration of the D8/ $\overline{D8}$ -branes,  $x_4$  is a function of  $y$ , we obtain the following behavior of the imaginary Lagrangian in the small temperature limit  $\chi \rightarrow 0$ ,

$$\begin{aligned}
\text{Im}\mathcal{L} &\simeq - \frac{2^3\pi^2 (2\pi\alpha')^{7/3} R^5 T_8}{3g_s} \vec{E} \cdot \vec{B} (1 + \zeta_T)^{-1/2} \mathcal{G}(Y_0) \\
&\times \left\{ 1 + \chi^{1/3} Y_0 \left[ 1 - Y_0^3 \zeta_T (1 + \zeta_T)^3 \chi \right]^{4/3} \left[ 1 - Y_0^3 (1 + \zeta_T)^4 \chi \right] x_4'^2(Y_0) \right\}^{1/2} \varepsilon \\
&= - \frac{2^3\pi^2 (2\pi\alpha')^{7/3} R^5 T_8}{3g_s} \vec{E} \cdot \vec{B} (1 + \zeta_T)^{-1/2} \mathcal{G}(Y_0) \\
&\times \left\{ \varepsilon^2 + \chi^{1/3} Y_0 \left[ 1 - Y_0^3 \zeta_T (1 + \zeta_T)^3 \chi \right]^{4/3} \left[ 1 - Y_0^3 (1 + \zeta_T)^4 \chi \right] [x_4(Y_0 + \varepsilon) - x_4(Y_0)]^2 \right\}^{1/2} \\
&= \text{finite}, \tag{4.22}
\end{aligned}$$

where

$$\mathcal{G}(Y) = Y^{-3/2} \left[ 1 - Y^3 \zeta_T (1 + \zeta_T)^3 \chi \right]^{-5/6} \left[ 1 - Y^3 \zeta_T (1 + \zeta_T)^4 \chi \right]^{-1/2}. \tag{4.23}$$

Therefore according to the evaluation, the imaginary part of the effective Lagrangian is always finite at zero temperature limit. We have also confirmed our conclusion numerically with arbitrary temperature. It would be very interesting to compare our calculations in this section with the D3/D7 approach in [13, 14]. The creation of quark-antiquark is proportional to  $\log \frac{1}{T}$  in [13, 14] which diverges at zero temperature limit, while it is always finite in our D0-D4/D8 system or the original Witten-Sakai-Sugimoto model. However, this result is not surprised because there is a confining scale in the compactified D4-brane (with or without smeared D0-branes) system as (2.4) or (2.5), and on the other hand the bubble configuration would be thermodynamically dominated at low temperature while the black brane configuration arises at high temperature [22, 24, 25, 26]. Consequently our calculation in the black D0-D4 configuration (high temperature) consistently coincides with the case in bubble D0-D4 (low temperature) at zero temperature limit, which means the creation rate should be definitely finite. Besides, as the situation of bubble D0-D4 system, we also find an additionally possible instability based on our calculations in (4.16) since the first line in (4.16) may also be imaginary. So it is the instability from the vacuum without the electromagnetic field as discussed in the bubble case.

## 5 Summary

In this paper, we have derived the effective Euler-Heisenberg Lagrangian holographically, which is identified as the DBI action of the flavor branes with the constant electromagnetic fields. Then we explore the electromagnetic instability and evaluated the pair creation rate of quark-antiquark in the Schwinger effect. With the D0-D4/D8 model in string theory, our investigation contains the influence of the D0-brane density which could be interpreted as the effect from  $\theta$  angle or chiral potential in QCD. In the bubble configuration, since the D4-branes with smeared

D0-branes are wrapped on a cycle, it introduces a confining scale into this system. Therefore, we obtain a very different result from  $\mathcal{N} = 2$  SQCD which is studied by the D3/D7 approach [13, 14]. More interestingly, once taking into account the non-vanished number density of D0-branes, our calculation shows an additional instability from the vacuum, which thus is independent on the electromagnetic field. Since the number density of D0-branes could be identified as the  $\theta$  angle in the D0-D4/D8 system, we accordingly suggest that this vacuum instability might holographically describe the flavored vacuum decay by the  $\theta$  angle in large  $N_c$  QCD.

In order to investigate the electromagnetic instability, we have removed the vacuum instability by small  $\theta$  angle or  $\zeta$  expansion in our calculations. With this method, we obtain the  $\theta$ -dependent creation rate of quark-antiquark in the bubble D0-D4 background and this result exactly coincides with [12] if setting  $\theta$  angle or  $\zeta = 0$  (i.e. no D0-branes). Our numerical calculation shows the creation rate decreases when  $\theta$  angle or  $\zeta$  increases and its behavior depends on the direction of the magnetic field relative to the electric field. Furthermore, the creation rate in the black D0-D4 background has also been computed which remains to be a finite result as oppose to the D3/D7 approach [13, 14]. Nevertheless, our result is significant since at zero temperature limit, the creation rate obtained in the black D0-D4 background qualitatively returns to the result from the bubble D0-D4 background, which coincides with the Hawking-Page transition in the compactified D4-brane system consistently [22, 26].

However, there might be some issues concerning the instability of our holographic setup in this paper. For examples, first, our numerical calculations are all based on the small  $\theta$  angle or  $\zeta$  expansion, so it is natural to ask what if we keep all the orders of  $\theta$  angle or  $\zeta$ ? How the electromagnetic and vacuum instability would be affected? Second, the electromagnetic field is non-dynamical in our calculations. So what about a dynamical case? Unfortunately, as opposed to the situation of the black D0-D4 background, it seems impossible to introduce an electric current directly in the bubble D0-D4 configuration since the flavor branes never end in the bulk. To solve this problem, one may need a baryon vertex. But it is less clear whether or not such a baryon vertex could be created. We leave these issues to a future study.

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